

Symmetry of uniaxial global Landau-de Gennes minimizers in the theory of nematic liquid crystals

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1 Preliminaries

Let $B(0, R_0) \subset \mathbb{R}^3$ denote a three-dimensional spherical droplet of radius $R_0 > 0$, centered at the origin. Let S_0 denote the set of symmetric, traceless 3×3 matrices i.e.

$$S_0 = \{\mathbf{Q} \in M^{3 \times 3}; \mathbf{Q}_{ij} = \mathbf{Q}_{ji}; \mathbf{Q}_{ii} = 0\} \quad (1)$$

where $M^{3 \times 3}$ is the set of 3×3 matrices. The corresponding matrix norm is defined to be [12]

$$|\mathbf{Q}|^2 = \mathbf{Q}_{ij} \mathbf{Q}_{ij} \quad i, j = 1 \dots 3 \quad (2)$$

and we will use the Einstein summation convention throughout the paper.

We work with the Landau-de Gennes theory for nematic liquid crystals [7] whereby a liquid crystal configuration is described by a macroscopic order parameter, known as the \mathbf{Q} -tensor order parameter. Mathematically, the Landau-de Gennes \mathbf{Q} -tensor order parameter is a symmetric, traceless 3×3 matrix belonging to the space S_0 in (1). The liquid crystal energy is given by the Landau-de Gennes energy functional and the associated energy density is a nonlinear function of \mathbf{Q} and its spatial derivatives [7, 17]. We work with the simplest form of the Landau-de Gennes energy functional that allows for a first-order nematic-isotropic phase transition and spatial inhomogeneities as shown below [12] -

$$\mathbf{I}_{LG}[\mathbf{Q}] = \int_{B(0, R_0)} \frac{L}{2} |\nabla \mathbf{Q}|^2 + f_B(\mathbf{Q}) \, dV. \quad (3)$$

Here, $L > 0$ is a small material-dependent elastic constant, $|\nabla \mathbf{Q}|^2 = \mathbf{Q}_{ij,k} \mathbf{Q}_{ij,k}$ (note that $\mathbf{Q}_{ij,k} = \frac{\partial \mathbf{Q}_{ij}}{\partial x_k}$) with $i, j, k = 1 \dots 3$ is an *elastic energy density* and $f_B : S_0 \rightarrow \mathbb{R}$ is the *bulk energy density*. For our purposes, we take f_B to be a quartic polynomial in the \mathbf{Q} -tensor invariants as shown below -

$$f_B(\mathbf{Q}) = \frac{\alpha(T - T^*)}{2} \text{tr} \mathbf{Q}^2 - \frac{b^2}{3} \text{tr} \mathbf{Q}^3 + \frac{c^2}{4} (\text{tr} \mathbf{Q}^2)^2 \quad (4)$$

where $\text{tr} \mathbf{Q}^3 = \mathbf{Q}_{ij} \mathbf{Q}_{jp} \mathbf{Q}_{pi}$ with $i, j, p = 1 \dots 3$, $\alpha, b^2, c^2 > 0$ are material-dependent constants, T is the absolute temperature and T^* is a characteristic temperature below which the isotropic phase $\mathbf{Q} = 0$ loses its stability. We work in the low-temperature regime with $T \ll T^*$ and hence, we can re-write (4) as

$$f_B(\mathbf{Q}) = -\frac{a^2}{2} \text{tr} \mathbf{Q}^2 - \frac{b^2}{3} \text{tr} \mathbf{Q}^3 + \frac{c^2}{4} (\text{tr} \mathbf{Q}^2)^2 \quad (5)$$

where $a^2 > 0$ is a temperature-dependent parameter and we will subsequently investigate the $a^2 \rightarrow \infty$ limit, known as the *low-temperature* limit. One can readily verify that f_B is bounded from below and attains its minimum on the set of \mathbf{Q} -tensors given by [11, 14]

$$\mathbf{Q}_{\min} = \left\{ \mathbf{Q} \in S_0; \mathbf{Q} = s_+ \left(\mathbf{n} \otimes \mathbf{n} - \frac{\mathbf{I}}{3} \right), \mathbf{n} \in S^2 \right\} \quad (6)$$

where \mathbf{I} is the 3×3 identity matrix and

$$s_+ = \frac{b^2 + \sqrt{b^4 + 24a^2c^2}}{4c^2}. \quad (7)$$

We are interested in characterizing global minimizers of the Landau-de Gennes energy functional in (3), on spherical droplets with *homeotropic* or *radial anchoring conditions* [14]. The global Landau-de Gennes minimizers correspond to physically observable liquid crystal configurations and hence, are of both mathematical and practical importance. We take our admissible \mathbf{Q} -tensors to belong to the space

$$\mathcal{A} = \{ \mathbf{Q} \in W^{1,2}(B(0, R_0); S_0); \mathbf{Q} = \mathbf{Q}_b \text{ on } \partial B(0, R_0) \} \quad (8)$$

where $W^{1,2}(B(0, R_0); S_0)$ is the Sobolev space of square-integrable \mathbf{Q} -tensors with square-integrable first derivatives [5]. The Dirichlet boundary condition \mathbf{Q}_b is given by

$$\mathbf{Q}_b(\mathbf{x}) = s_+ \left(\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} - \frac{\mathbf{I}}{3} \right) \in \mathbf{Q}_{\min} \quad (9)$$

where $\mathbf{x} \in \mathbb{R}^3$ is the position vector and $\frac{\mathbf{x}}{|\mathbf{x}|}$ is the unit-vector in the radial direction. For completeness, we recall that the $W^{1,2}$ -norm is given by $\|\mathbf{Q}\|_{W^{1,2}} = \left(\int_{B(0, R_0)} |\mathbf{Q}|^2 + |\nabla \mathbf{Q}|^2 dV \right)^{1/2}$ and the L^∞ -norm is defined to be $\|\mathbf{Q}\|_{L^\infty} = \text{ess sup}_{\mathbf{x} \in B(0, R_0)} |\mathbf{Q}(\mathbf{x})|$ [5].

We define a modified Landau-de Gennes energy functional as shown below -

$$\hat{\mathbf{I}}_{\mathbf{LG}}[\mathbf{Q}] = \int_{B(0, R_0)} \frac{L}{2} |\nabla \mathbf{Q}|^2 + f_B(\mathbf{Q}) - \min_{\mathbf{Q} \in S_0} f_B(\mathbf{Q}) dV \quad (10)$$

where $f_B(\mathbf{Q}) - \min_{\mathbf{Q} \in S_0} f_B(\mathbf{Q}) \geq 0$ for all $\mathbf{Q} \in S_0$. The existence of a global minimizer for the modified functional $\hat{\mathbf{I}}_{\mathbf{LG}}$ in the admissible space \mathcal{A} is immediate from the direct methods in the calculus of variations [5]; the details are omitted for brevity. It is clear that $\mathbf{Q}^* \in \mathcal{A}$ is a minimizer of $\hat{\mathbf{I}}_{\mathbf{LG}}$ if and only if \mathbf{Q}^* is a minimizer of $\mathbf{I}_{\mathbf{LG}}$ in (3) and hence, it suffices to study minimizers of the modified functional in (10). In what follows, we study the modified functional in (10) and drop the *hat* for brevity.

For a fixed $a^2 > 0$, let $\mathbf{Q}^a \in \mathcal{A}$ denote a corresponding Landau-de Gennes global minimizer. The Euler-Lagrange equations associated with $\hat{\mathbf{I}}_{\mathbf{LG}}$ are given by a nonlinear elliptic system of coupled partial differential equations:

$$L\Delta \mathbf{Q}_{ij} = -a^2 \mathbf{Q}_{ij} - b^2 \left(\mathbf{Q}_{ip} \mathbf{Q}_{pj} - \frac{1}{3} \mathbf{Q}_{pq} \mathbf{Q}_{pq} \delta_{ij} \right) + c^2 (\text{tr} \mathbf{Q}^2) \mathbf{Q}_{ij} \quad i, j, p, q = 1 \dots 3 \quad (11)$$

where $\frac{b^2}{3} \mathbf{Q}_{pq} \mathbf{Q}_{pq} \delta_{ij}$ is a Lagrange multiplier accounting for the tracelessness constraint [12]. It follows from standard arguments in elliptic regularity that \mathbf{Q}^a is smooth and real analytic on $B(0, R_0)$.

The notion of a *limiting harmonic map* was first introduced in [12] and is crucial in what follows. A limiting harmonic map $\mathbf{Q}^0 \in \mathcal{A}$ is defined to be

$$\mathbf{Q}^0 = s_+ \left(\mathbf{n}^0 \otimes \mathbf{n}^0 - \frac{\mathbf{I}}{3} \right) \quad (12)$$

where \mathbf{n}^0 is a minimizer of the Dirichlet energy [19]

$$I[\mathbf{n}] = \int_{B(0, R_0)} |\nabla \mathbf{n}|^2 dV \quad (13)$$

in the admissible space $\mathcal{A}_{\mathbf{n}} = \left\{ \mathbf{n} \in W^{1,2}(B(0, R_0); S^2); \mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|} \text{ on } \partial B(0, R_0) \right\}$. In the case of a spherical droplet with homeotropic boundary conditions, \mathbf{n}^0 is unique and given by the radial unit-vector [10]. Hence, the limiting harmonic map is unique for our model problem and is given by

$$\mathbf{Q}^0 = s_+ \left(\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} - \frac{\mathbf{I}}{3} \right). \quad (14)$$

We note that \mathbf{Q}^0 has a single isolated point defect at the origin.

In what follows, we keep L, b^2 and c^2 fixed in (10). Following the methods in [14], we first introduce a re-scaling relevant to the *low-temperature* limit: $a^2 \rightarrow \infty$. Let t denote a *dimensionless* temperature

$$t = \frac{27a^2c^2}{b^4} > 0 \quad (15)$$

so that the $a^2 \rightarrow \infty$ limit corresponds to the $t \rightarrow \infty$ limit and define

$$h_+ = \frac{3 + \sqrt{9 + 8t}}{4} \sim \sqrt{\frac{t}{2}} \quad t \rightarrow \infty. \quad (16)$$

Define

$$\bar{\mathbf{Q}}_{ij} = \frac{1}{h_+} \sqrt{\frac{27c^4}{2b^4}} \mathbf{Q}_{ij} \quad (17)$$

and the corresponding Landau-de Gennes energy functional is given by (up to a multiplicative constant)

$$\hat{\mathbf{I}}_{\mathbf{LG}}[\bar{\mathbf{Q}}] = \int_{B(0, R_0)} \frac{\bar{L}}{2} |\nabla \bar{\mathbf{Q}}|^2 + \frac{t}{8} \left[(1 - |\bar{\mathbf{Q}}|^2)^2 - \frac{8h_+}{t} \sqrt{\frac{3}{2}} \text{tr} \bar{\mathbf{Q}}^3 + f(t) \right] dV \quad (18)$$

where $\bar{L} = \frac{27c^2L}{2b^4} > 0$ is fixed, $f(t)$ is a function that can be explicitly computed,

$(1 - |\bar{\mathbf{Q}}|^2)^2 - \frac{8h_+}{t} \sqrt{\frac{3}{2}} \text{tr} \bar{\mathbf{Q}}^3 + f(t) \geq 0$ for $\mathbf{Q} \in S_0$ (see definition of $\hat{\mathbf{I}}_{\mathbf{LG}}$ in (10)) and for t sufficiently large, we have

$$\frac{\sigma_1}{\sqrt{t}} \leq \frac{h_+}{t} \leq \frac{\sigma_2}{\sqrt{t}}; \quad \frac{\gamma_1}{\sqrt{t}} \leq f(t) \leq \frac{\gamma_2}{\sqrt{t}} \quad t \rightarrow \infty \quad (19)$$

for positive constants σ_1, σ_2 and constants γ_1, γ_2 independent of t in the $t \rightarrow \infty$ limit. The corresponding admissible space is

$$\bar{\mathcal{A}} = \left\{ \bar{\mathbf{Q}} \in W^{1,2}(B(0, R_0); S_0); \bar{\mathbf{Q}} = \frac{1}{h_+} \sqrt{\frac{27c^4}{2b^4}} \mathbf{Q}_b \text{ on } \partial B(0, R_0) \right\} \quad (20)$$

where \mathbf{Q}_b has been defined above. In what follows, we drop the *bars* from the re-scaled variables for brevity and all subsequent statements in this section are to be understood in terms of the scaled variable in (17).

Next, we quote some results from [12] that are relevant to the development of our mathematical framework.

Proposition 1 *For each $t > 0$, let $\mathbf{Q}^t = s^t (\mathbf{n}^t \otimes \mathbf{n}^t - \frac{1}{3}\mathbf{I})$ denote a uniaxial global minimizer of $\hat{\mathbf{I}}_{\mathbf{LG}}$ in (18), in the admissible space $\bar{\mathcal{A}}$ defined in (20), where $s^t \in W^{1,2}(\Omega, \mathbb{R})$ and $\mathbf{n}^t \in W^{1,2}(\Omega, S^2)$. Then there exists a sequence $\{t_k\}$ with $t_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $\mathbf{Q}^{t_k} \rightarrow \mathbf{Q}^0$ strongly in $W^{1,2}(B(0, R_0); S_0)$, where \mathbf{Q}^0 is the re-scaled limiting harmonic map defined in (14).*

Proof: The proof of Proposition 1 is identical to the proof of Lemma 3 in [12] and the details are omitted for brevity. From the global energy minimality, we have

$$\hat{\mathbf{I}}_{\mathbf{LG}}[\mathbf{Q}^{t_k}] \leq \hat{\mathbf{I}}_{\mathbf{LG}}[\mathbf{Q}^0] = E(L, R_0) \quad \forall k \quad (21)$$

where $E > 0$ is a positive constant independent of t_k i.e. the \mathbf{Q}^{t_k} 's have bounded energy in the limit $k \rightarrow \infty$. Note that

$$\frac{t}{8} \left[(1 - |\mathbf{Q}^0|^2)^2 - \frac{8h_+}{t} \sqrt{\frac{3}{2}} \text{tr}(\mathbf{Q}^0)^3 + f(t) \right] = 0$$

for all $t > 0$, by definition (since $\mathbf{Q}^0 \in \mathbf{Q}_{\min}$). One immediate consequence of the strong convergence result are the following integral equalities -

$$\begin{aligned} \int_{B(0, R_0)} |\nabla \mathbf{Q}^0|^2 dV &= \lim_{k \rightarrow \infty} \int_{B(0, R_0)} |\nabla \mathbf{Q}^{t_k}|^2 dV \\ \lim_{k \rightarrow \infty} \int_{B(0, R_0)} \left[(1 - |\mathbf{Q}^{t_k}|^2)^2 - \frac{8h_+}{t} \sqrt{\frac{3}{2}} \text{tr}(\mathbf{Q}^{t_k})^3 + f(t) \right] dV &= 0. \end{aligned} \quad (22)$$

From the inequalities (19), we deduce that

$$\lim_{k \rightarrow \infty} \int_{B(0, R_0)} (1 - |\mathbf{Q}^{t_k}|^2)^2 dV = 0. \quad (23)$$

□

Proposition 2 *For each $t > 0$, let \mathbf{Q}^t denote a global minimizer of $\hat{\mathbf{I}}_{\mathbf{LG}}$ in (18), in the admissible space $\bar{\mathcal{A}}$ defined in (20). Let $\{t^k\}$ be a sequence such that $t^k \rightarrow \infty$ as $k \rightarrow \infty$. Then*

$$\lim_{k \rightarrow \infty} |\mathbf{Q}^{t_k}(\mathbf{x})| \leq 1 \quad \forall \mathbf{x} \in B(0, R_0). \quad (24)$$

Proof: The proof follows from a standard maximum principle; see [12] Proposition 3 for an analogous statement with proof. □

Proposition 3 *[[12]; Lemma 2] For each $t > 0$, let \mathbf{Q}^t denote a global minimizer of $\hat{\mathbf{I}}_{\mathbf{LG}}$ in (18), in the admissible space $\bar{\mathcal{A}}$ defined in (20). Define $B(\mathbf{x}, r) = \{\mathbf{y} \in B(0, R_0); |\mathbf{x} - \mathbf{y}| \leq r\} \subset B(0, R_0)$ and*

$$e(\mathbf{Q}^t, \nabla \mathbf{Q}^t) = \frac{\bar{L}}{2} |\nabla \mathbf{Q}^t|^2 + \frac{t}{8} \left[(1 - |\mathbf{Q}^t|^2)^2 - \frac{8h_+}{t} \sqrt{\frac{3}{2}} \text{tr}(\mathbf{Q}^t)^3 + f(t) \right].$$

Then

$$\frac{1}{r} \int_{B(\mathbf{x}, r)} e(\mathbf{Q}^t, \nabla \mathbf{Q}^t) dV \leq \frac{1}{R} \int_{B(\mathbf{x}, R)} e(\mathbf{Q}^t, \nabla \mathbf{Q}^t) dV, \quad \forall \mathbf{x} \in B(0, R_0), \quad r \leq R, \quad (25)$$

so that $B(\mathbf{x}, R) \subset B(0, R_0)$.

Proof: The proof can be found in [[12]; Lemma 2]. An analogous boundary monotonicity formula can be found in Lemma 9 [12]. \square

Proposition 4 *For each $t > 0$, let \mathbf{Q}^t denote a global uniaxial minimizer of $\hat{\mathbf{I}}_{\mathbf{LG}}$ in (18), in the admissible space $\bar{\mathcal{A}}$ defined in (20). Let $\{t^k\}$ be a sequence such that $t^k \rightarrow \infty$ as $k \rightarrow \infty$ with $\mathbf{Q}^{t_k} \rightarrow \mathbf{Q}^0$ in $W^{1,2}(B(0, R_0), S_0)$ as $k \rightarrow \infty$, where \mathbf{Q}^0 is the re-scaled limiting harmonic map defined in (14). For any compact $K \subset B(0, R_0)$ such that K does not contain any singularities of \mathbf{Q}^0 i.e. does not contain the origin, we have*

$$\lim_{k \rightarrow \infty} |\mathbf{Q}^{t_k}(\mathbf{x})| = 1 \quad \forall \mathbf{x} \in K \quad (26)$$

and the limit is uniform on K .

Proof: Proposition 4 is a consequence of the pointwise uniform convergence

$$\lim_{k \rightarrow \infty} \left[(1 - |\mathbf{Q}^{t_k}|)^2 - \frac{8h_+}{t} \sqrt{\frac{3}{2}} \text{tr}(\mathbf{Q}^{t_k})^3 + f(t_k) \right] = 0$$

everywhere away from the singular set of \mathbf{Q}^0 i.e. away from the origin. This uniform convergence result holds in the interior and up to the boundary. The proof can be found in [12], Propositions 4 and 6. \square

To summarize, let $\{t^k\}$ be a sequence such that $t^k \rightarrow \infty$ as $k \rightarrow \infty$ and let $\{\mathbf{Q}^{t_k}\}$ denote a corresponding sequence of purely uniaxial Landau-de Gennes minimizers. Then up to a subsequence, $\mathbf{Q}^{t_k} \rightarrow \mathbf{Q}^0$ in $W^{1,2}(B(0, R_0), S_0)$ as $k \rightarrow \infty$. Further for k sufficiently large, $\hat{\mathbf{I}}_{\mathbf{LG}}[\mathbf{Q}^{t_k}]$ can be bounded independently of t_k since

$$\hat{\mathbf{I}}_{\mathbf{LG}}[\mathbf{Q}^{t_k}] \leq \hat{\mathbf{I}}_{\mathbf{LG}}[\mathbf{Q}^0]$$

and $|\mathbf{Q}^{t_k}|$ is strictly positive (and bounded from below) everywhere away from the origin.

2 Statement of main results

Our main result is the following :

Theorem 1 *Let $B(0, R_0) \subset \mathbb{R}^3$ denote a spherical droplet of radius, R_0 , centered at the origin. For each $a^2 > 0$, let \mathbf{Q}^a denote a global Landau-de Gennes minimizer in the admissible space \mathcal{A} defined in (8). Then in the limit $a^2 \rightarrow \infty$, \mathbf{Q}^a cannot be purely uniaxial i.e. cannot be of the form*

$$\mathbf{Q}^a(\mathbf{x}) = s(\mathbf{x}) \left(\mathbf{n} \otimes \mathbf{n} - \frac{\mathbf{I}}{3} \right) \quad \mathbf{x} \in B(0, R_0) \quad (27)$$

for a function $s : B(0, R_0) \rightarrow \mathbb{R}$ and a unit-vector field $\mathbf{n} \in W^{1,2}(B(0, R_0), S^2)$.

Comment: From [1], it is known that if $\mathbf{Q} = s(\mathbf{n} \otimes \mathbf{n} - \frac{\mathbf{I}}{3}) \in W^{1,2}(\Omega, S_0)$ on a simply-connected domain $\Omega \subset \mathbb{R}^3$, then $\mathbf{n} \in W^{1,2}(\Omega, S^2)$.

We prove Theorem 1 by contradiction in the subsequent sections. Of key importance is the division trick used in [18] for the Ginzburg-Landau theory for superconductivity in three dimensions. We adapt the division trick in [18] to the Landau-de Gennes framework for nematic liquid crystals. We point out the following important differences: (i) we have a parameter a^2 and we are interested in the asymptotics of global energy minimizers in the $a^2 \rightarrow \infty$ limit; the Ginzburg-Landau equations in [18] are parameter-free, (ii) the nonlinearities in the Landau-de Gennes equations (11) are different to the nonlinearities in the Ginzburg-Landau equations introducing additional technical complexities and (iii) the Landau-de Gennes macroscopic variable is a two-tensor field $\mathbf{Q} \in S_0$ whereas the Ginzburg-Landau macroscopic variable in \mathbb{R}^3 is a three-dimensional vector field $\mathbf{u} \in \mathbb{R}^3$.

To make better contact with the framework used in [18], we introduce the following dimensionless variables as in [14] :

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{\xi_b}, \quad \tilde{\mathbf{Q}} = \frac{1}{h_+} \sqrt{\frac{27c^4}{2b^4}} \mathbf{Q}, \quad \tilde{\mathcal{I}}_{LG} = \frac{h_+^2}{\sqrt{t}} \sqrt{\left(\frac{27c^6}{4b^4 L^3}\right)} \hat{\mathbf{I}}_{LG} \quad (28)$$

where $t = \frac{27a^2 c^2}{b^4} > 0$ is the *reduced temperature* [15], $t > 1$ throughout the paper and h_+ has been defined in (16). The length-scale $\xi_b = \sqrt{\frac{27c^2 L}{tb^4}}$. We note that the position vector \mathbf{x} has been re-scaled in (28) whereas it was left unchanged in (17). The corresponding dimensionless energy density is

$$\tilde{e}(\tilde{\mathbf{Q}}, \nabla \tilde{\mathbf{Q}}) = \frac{1}{2} |\nabla \tilde{\mathbf{Q}}|^2 - \frac{1}{2} \text{tr} \tilde{\mathbf{Q}}^2 - \frac{\sqrt{6} h_+}{t} \text{tr} \tilde{\mathbf{Q}}^3 + \frac{h_+^2}{2t} (\text{tr} \tilde{\mathbf{Q}}^2)^2 + C(t) \quad (29)$$

where $C(t) = \frac{1}{2} + \frac{h_+}{t} - \frac{h_+^2}{2t}$ is an additive constant that ensures

$$-\frac{1}{2} \text{tr} \tilde{\mathbf{Q}}^2 - \frac{\sqrt{6} h_+}{t} \text{tr} \tilde{\mathbf{Q}}^3 + \frac{h_+^2}{2t} (\text{tr} \tilde{\mathbf{Q}}^2)^2 + C(t) \geq 0 \quad \forall \mathbf{Q} \in S_0.$$

The corresponding Landau-de Gennes energy functional is given by

$$\tilde{\mathcal{I}}_{LG}[\tilde{\mathbf{Q}}] = \int_{B(0, \tilde{R}_t)} \tilde{e}(\tilde{\mathbf{Q}}, \nabla \tilde{\mathbf{Q}}) dV, \quad (30)$$

where $\tilde{R}_t = \gamma \sqrt{t} R_0$ for a fixed constant $\gamma > 0$. In particular, $\tilde{R}_t \rightarrow \infty$ as $t \rightarrow \infty$. In what follows, we drop the *tilde* on the dimensionless variables for brevity and all subsequent results are to be understood in terms of the dimensionless variables. From (8) and (20), the admissible \mathbf{Q} -tensors belong to the space

$$\mathcal{A}_{\mathbf{Q}} = \left\{ \mathbf{Q} \in W^{1,2}(B(0, R_t), S_0); \mathbf{Q} = \sqrt{\frac{3}{2}} \left(\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} - \frac{1}{3} \mathbf{I} \right) \text{ on } \partial B(0, R_t) \right\} \quad (31)$$

The associated Euler-Lagrange equations are [13, 12] -

$$\Delta \mathbf{Q}_{ij} = -\mathbf{Q}_{ij} - \frac{3\sqrt{6} h_+}{t} \left(\mathbf{Q}_{ik} \mathbf{Q}_{kj} - \frac{\delta_{ij}}{3} \text{tr}(\mathbf{Q}^2) \right) + \frac{2h_+^2}{t} \mathbf{Q}_{ij} \text{tr}(\mathbf{Q}^2), \quad i, j = 1, 2, 3. \quad (32)$$

All global and local energy minimizers in $\mathcal{A}_{\mathbf{Q}}$ are classical solutions of (32).

Proposition 5 For each $t > 0$, assume that a uniaxial global minimizer of $\tilde{\mathcal{I}}_{LG}$ exists in the admissible space $\mathcal{A}_{\mathbf{Q}}$ defined in (31); we denote this uniaxial minimizer by \mathbf{Q}^t . Let $\{t^k\}$ be a sequence such that $t^k \rightarrow \infty$ as $k \rightarrow \infty$ with $\mathbf{Q}^{t_k} \rightarrow \mathbf{Q}^0$ in $W^{1,2}(B(0, R_{t_k}), S_0)$ as $k \rightarrow \infty$, where \mathbf{Q}^0 is the limiting harmonic map

$$\mathbf{Q}^0 = \sqrt{\frac{3}{2}} \left(\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} - \frac{\mathbf{I}}{3} \right).$$

Then from Proposition 4,

$$\lim_{t_k \rightarrow \infty} \int_{B(0, R_{t_k})} -\frac{1}{2} \text{tr} \tilde{\mathbf{Q}}^2 - \frac{\sqrt{6}h_+}{t_k} \text{tr} \tilde{\mathbf{Q}}^3 + \frac{h_+^2}{2t_k} (\text{tr} \tilde{\mathbf{Q}}^2)^2 + C(t_k) = 0 \quad (33)$$

where $R_{t_k} \propto \sqrt{t_k}$. For k sufficiently large, we have

(i) $\mathbf{Q}^{t_k} = \sqrt{\frac{3}{2}} |\mathbf{Q}^{t_k}| (\mathbf{n} \otimes \mathbf{n} - \frac{1}{3} \mathbf{I})$ for some $\mathbf{n} \in W^{1,2}(B(0, R_{t_k}), S^2)$.

(ii) \mathbf{Q}^{t_k} is a classical solution of the following nonlinear system of elliptic partial differential equations on $B(0, R_{t_k})$ where $R_{t_k} \rightarrow \infty$ as $t_k \rightarrow \infty$:

$$\Delta \mathbf{Q}_{ij}^{t_k} = (|\mathbf{Q}^{t_k}|^2 - 1) \mathbf{Q}_{ij}^{t_k} + \frac{3h_+}{t_k} (|\mathbf{Q}^{t_k}|^2 - |\mathbf{Q}^{t_k}|) \mathbf{Q}_{ij}^{t_k} \quad (34)$$

(iii) $|\mathbf{Q}^{t_k}|(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in B(0, R_{t_k})$,

(iv) $\frac{1}{R_{t_k}} \tilde{\mathcal{I}}_{LG}[\mathbf{Q}^{t_k}] \leq 12\pi$ and

(v) $\lim_{t_k \rightarrow \infty} \mathbf{Q}^{t_k}(0) = 0$.

(vi) All derivatives of \mathbf{Q}^{t_k} can be bounded independently of t_k in the $k \rightarrow \infty$ limit i.e.

$$\|\nabla^j \mathbf{Q}^{t_k}\|_{L^\infty(B(0, R_{t_k}))} \leq C_j \quad j \geq 1 \quad (35)$$

for a positive constant C_j independent of t_k .

Proof: In what follows, we drop the subscript k for brevity and work in the $t \rightarrow \infty$ limit. *Proof of (i):* This follows from the uniaxial character of \mathbf{Q}^t i.e. it has two equal eigenvalues

$$\mathbf{Q}^t(\mathbf{x}) = \lambda(\mathbf{x}) (\mathbf{e}(\mathbf{x}) \otimes \mathbf{e}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) \otimes \mathbf{f}(\mathbf{x})) - 2\lambda \mathbf{n}(\mathbf{x}) \otimes \mathbf{n}(\mathbf{x})$$

where $\mathbf{e}, \mathbf{f}, \mathbf{n}$ form an orthonormal frame at \mathbf{x} and $\mathbf{n} \otimes \mathbf{n} + \mathbf{e} \otimes \mathbf{e} + \mathbf{f} \otimes \mathbf{f} = \mathbf{I}$. Using the above, \mathbf{Q}^t can be written in the simpler form

$$\mathbf{Q}^t(\mathbf{x}) = -3\lambda(\mathbf{x}) \left(\mathbf{n}(\mathbf{x}) \otimes \mathbf{n}(\mathbf{x}) - \frac{1}{3} \mathbf{I} \right) \quad (36)$$

where

$$|\mathbf{Q}^t|^2(\mathbf{x}) = 6\lambda^2(\mathbf{x}). \quad (37)$$

From [11], a global uniaxial Landau-de Gennes minimizer has $\lambda < 0$ and from (36)-(37),

$-3\lambda = \sqrt{\frac{3}{2}} |\mathbf{Q}^t|$. The representation formula in (i) follows. \square

Proof of (ii): If \mathbf{Q}^t is a uniaxial global Landau-de Gennes minimizer, then it is a classical solution of (32). The partial differential equations (34) follow from substituting the representation formula in (i) into (32). \square

Proof of (iii): The proof follows from multiplying both sides of (34) by $\mathbf{Q}_{ij}^{t_k}$ and applying a standard maximum principle argument for $|\mathbf{Q}^t|^2$; the details are omitted for brevity. \square

Proof of (iv): This is a direct consequence of the global energy minimality. The limiting harmonic map \mathbf{Q}^0 is simply given by

$$\mathbf{Q}^0 = \sqrt{\frac{3}{2}} \left(\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} - \frac{1}{3} \mathbf{I} \right) \quad (38)$$

in terms of the dimensionless variables in (28). A direct computation shows that $\tilde{\mathcal{I}}_{LG}[\mathbf{Q}^0] = 12\pi R_t$ (since $-\frac{1}{2}\text{tr}(\mathbf{Q}^0)^2 - \frac{\sqrt{6}h_+}{t}\text{tr}(\mathbf{Q}^0)^3 + \frac{h_+^2}{2t}(\text{tr}(\mathbf{Q}^0)^2)^2 + C(t) = 0$ by definition because $\mathbf{Q}^0 \in \mathbf{Q}_{\min}$) and hence

$$\tilde{\mathcal{I}}_{LG}[\mathbf{Q}^t] \leq \tilde{\mathcal{I}}_{LG}[\mathbf{Q}^0] = 12\pi R_t. \quad (39)$$

The inequality in (iv) follows. \square

Proof of (v): This follows from Proposition 4. We have a topologically non-trivial boundary condition \mathbf{Q}_b in (9) and hence every interior extension of $\frac{\mathbf{x}}{|\mathbf{x}|}$ must have interior discontinuities. The extension \mathbf{n} in (i) has interior discontinuities and at every such point of discontinuity \mathbf{x}^* , $\mathbf{Q}^t(\mathbf{x}^*) = 0$ (see [14] for further discussion on these lines; \mathbf{Q}^t is analytic at \mathbf{x}^* whereas \mathbf{n} is not and \mathbf{n} can lose regularity only when the number of distinct eigenvalues of \mathbf{Q}^t changes. The number of distinct eigenvalues of \mathbf{Q}^t can change only when \mathbf{Q}^t relaxes into the isotropic phase i.e. $\mathbf{Q}^t(\mathbf{x}^*) = 0$.) From Proposition 4, as $t \rightarrow \infty$, all isotropic points are concentrated near the singular set of \mathbf{Q}^0 and the singular set of \mathbf{Q}^0 merely consists of the origin. Hence, we have $\lim_{t \rightarrow \infty} \mathbf{Q}^t(0) = 0$. \square

Lemma 1 *Assume that \mathbf{Q}^t is a uniaxial global Landau-de Gennes minimizer in the admissible space $\mathcal{A}_{\mathbf{Q}}$ on the droplet $B(0, R_t)$ in the limit $t \rightarrow \infty$. Then the bulk energy density satisfies the following inequality -*

$$f(\mathbf{Q}^t) = -\frac{1}{2}\text{tr}(\mathbf{Q}^t)^2 - \frac{\sqrt{6}h_+}{t}\text{tr}(\mathbf{Q}^t)^3 + \frac{h_+^2}{2t}(\text{tr}(\mathbf{Q}^t)^2)^2 + \frac{1}{2} + \frac{h_+}{t} - \frac{h_+^2}{2t} \geq \frac{(1 - |\mathbf{Q}^t|^2)^2}{4}. \quad (40)$$

Proof: From Proposition 5, we have $0 \leq |\mathbf{Q}^t| \leq 1$ and one can check from the representation formula in Proposition 5 (i) that

$$\text{tr}(\mathbf{Q}^t)^3 = \frac{|\mathbf{Q}^t|^3}{\sqrt{6}}.$$

Substitute the above into the definition of $f(\mathbf{Q}^t)$ in (40)

$$f(\mathbf{Q}^t) = -\frac{1}{2}|\mathbf{Q}^t|^2 - \frac{h_+}{t}|\mathbf{Q}^t|^3 + \frac{h_+^2}{2t}|\mathbf{Q}^t|^4 + \frac{1}{2} + \frac{h_+}{t} - \frac{h_+^2}{2t} \quad (41)$$

and one can check from (41) that

$$f(\mathbf{Q}^t) \geq \frac{(1 - |\mathbf{Q}^t|^2)^2}{4}.$$

\square

Our second main result concerns the characterization of uniaxial global Landau-de Gennes minimizers if they exist.

Theorem 2 *Assume that $\mathbf{Q}^t \in \mathcal{A}_{\mathbf{Q}}$ is a uniaxial global Landau-de Gennes minimizer on the droplet $B(0, R_t)$ in the limit $t \rightarrow \infty$, where $R_t \propto \sqrt{t} \rightarrow \infty$ as $t \rightarrow \infty$. Then \mathbf{Q}^t is an entire solution of*

$$\Delta \mathbf{Q}_{ij}^t = (|\mathbf{Q}^t|^2 - 1) \mathbf{Q}_{ij}^t + \frac{3h_+}{t} (|\mathbf{Q}^t|^2 - |\mathbf{Q}^t|) \mathbf{Q}_{ij}^t \quad (42)$$

with $\mathbf{Q}^t(0) = 0$ and $\lim_{t \rightarrow \infty} \frac{1}{R_t} \tilde{\mathcal{L}}_{LG}[\mathbf{Q}^t] \leq 12\pi$. There exists a $\mathbf{T} \in \mathcal{O}(3)$ such that

$$\mathbf{Q}^t(\mathbf{x}) = h(|\mathbf{x}|) \left[\frac{\mathbf{T}\mathbf{x} \otimes \mathbf{T}\mathbf{x}}{|\mathbf{x}|^2} - \frac{1}{3}\mathbf{I} \right] \quad (43)$$

where $h : [0, \infty) \rightarrow \mathbb{R}^+$ is the unique, monotonically increasing solution of the ordinary differential equation

$$\frac{d^2 h}{dr^2} + \frac{2}{r} \frac{dh}{dr} - \frac{6h}{r^2} = h^3 - h + \frac{3h_+}{t} (h^3 - h^2) \quad (44)$$

(with $r = |\mathbf{x}|$) subject to the boundary conditions

$$\begin{aligned} h(0) &= 0 & h(r) &\rightarrow 1 & r &\rightarrow \infty \\ 0 &\leq h(r) \leq 1 \end{aligned} \quad (45)$$

in the limit $t \rightarrow \infty$.

The last ingredient of the proof of Theorem 1 is the following result from [15, 14]:

Theorem 3 [15, 14]: *The radial-hedgehog solution*

$$\mathbf{H} = h(|\mathbf{x}|) \left[\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} - \frac{1}{3}\mathbf{I} \right] \quad (46)$$

where h is the unique solution of (44) subject to (45), is a solution of the Euler-Lagrange equations (34). Moreover, \mathbf{H} is not a global Landau-de Gennes minimizer in the admissible space $\mathcal{A}_{\mathbf{Q}}$ in the limit $t \rightarrow \infty$.

Proof: One can explicitly construct a biaxial perturbation of the form

$$\mathbf{H}_b(\mathbf{x}) = \mathbf{H}(\mathbf{x}) + \frac{1}{(r^2 + 12)^2} \left(1 - \frac{r}{\sigma} \right) \left(\mathbf{z} \otimes \mathbf{z} - \frac{\mathbf{I}}{3} \right)$$

with $\sigma = 10$ and $\mathbf{z} = (0, 0, 1)$, that has lower Landau-de Gennes energy than the radial-hedgehog solution in the limit $t \rightarrow \infty, R_t \rightarrow \infty$. This perturbation does not have an isotropic core at the origin. The details can be found in [14, 15]. \square

To summarize, we obtain symmetry results for global uniaxial Landau-de Gennes minimizers in the limit $t \rightarrow \infty$ i.e. we show that every uniaxial Landau-de Gennes minimizer is the radial-hedgehog solution in (46) (modulo a rotation) in the limit $t \rightarrow \infty$. The final step is to use results on the radial-hedgehog solution in the limit $t \rightarrow \infty$ from [15, 14] to prove Theorem 1. In the subsequent sections, we proceed with the proof of Theorem 2.

3 Symmetry of uniaxial Landau-de Gennes minimizers

The main trick is to adapt the division trick of [18] to the Landau-de Gennes framework for nematic liquid crystals. Let \mathbf{Q}^t denote a global uniaxial Landau-de Gennes minimizer in the space $\mathcal{A}_{\mathbf{Q}}$ for t sufficiently large so that Proposition 5 holds. Define

$$\mathbf{S}_{ij}(\mathbf{x}) = \frac{\mathbf{Q}_{ij}^t(\mathbf{x})}{h(|\mathbf{x}|)} \quad (47)$$

where h is the unique solution of (44) subject to (45) in the $t \rightarrow \infty$ limit. We first prove some auxiliary properties of the function $h : [0, \infty) \rightarrow [0, 1]$ in (44).

Theorem 4 Define the radial-hedgehog solution \mathbf{H} as in (46); \mathbf{H} is a solution of the system of partial differential equations (34) for all $t > 1$. In the limit $t \rightarrow \infty$,
(i) $h : [0, \infty) \rightarrow [0, 1)$ is analytic;

$$h(0) = h'(0) = 0; h''(0) > 0; \quad (48)$$

$\mathbf{H}_{ij}(0) = 0$ and $\nabla \mathbf{H}_{ij}(0) = 0$,
(ii) $0 \leq h(r) \leq 1$ and

$$\frac{r^2}{r^2 + 14} \leq h(r) \leq \frac{r^2}{r^2 + t\lambda_t^2} \quad (49)$$

where $t\lambda_t^2 \leq \frac{3}{t}$,

(iii) we have the following gradient bound

$$|h'(|\mathbf{x}|)| \leq C_1 \|\nabla \mathbf{H}\|_{L^\infty(\mathbb{R}^3)} \leq C_2 \quad (50)$$

where C_1 and C_2 are positive constants independent of t as $t \rightarrow \infty$,

(iv) we have the following bound for the second derivative

$$|h''(|\mathbf{x}|)| \leq C_3 \|\nabla^2 \mathbf{H}\|_{L^\infty(\mathbb{R}^3)} \leq C_4 \quad (51)$$

where C_3 and C_4 are positive constants independent of t as $t \rightarrow \infty$ and

(v) we have the following bound for the third derivative (noting that $\left|\frac{\mathbf{H}_{ij}}{h}\right| = 1$)

$$|h'''(|\mathbf{x}|)| = \left| \frac{\mathbf{H}_{ij} \mathbf{H}_{ij, \alpha\beta\gamma} \mathbf{x}_\alpha \mathbf{x}_\beta \mathbf{x}_\gamma}{h} \right| \leq C_5 \|\nabla^3 \mathbf{H}\|_{L^\infty(\mathbb{R}^3)} \leq C_6 \quad (52)$$

where C_5 and C_6 are positive constants independent of t as $t \rightarrow \infty$.

Proof of (i): The analyticity of h , the relations (48) and $\mathbf{H}_{ij}(0) = 0$ have been proven in [14]. To prove $\nabla \mathbf{H}_{ij}(0) = 0$, we use the following equality

$$\mathbf{H}_{ij}(\mathbf{x}) \mathbf{H}_{ij}(\mathbf{x}) = h^2(|\mathbf{x}|), \quad i, j = 1 \dots 3 \quad (53)$$

so that for any fixed direction characterized by the unit-vector \mathbf{e}_α where $\alpha = 1 \dots 3$, we have

$$\mathbf{H}_{ij}(\mathbf{x}) \mathbf{H}_{ij, \alpha}(\mathbf{x}) = h(|\mathbf{x}|) h'(|\mathbf{x}|) \frac{\mathbf{x}_\alpha}{|\mathbf{x}|}$$

where $\mathbf{x}_\alpha = \mathbf{x} \cdot \mathbf{e}_\alpha$. We set $\mathbf{x} = |\mathbf{x}| \mathbf{e}_\alpha$ and re-write the above as

$$\lim_{|\mathbf{x}| \rightarrow 0} \frac{\mathbf{H}_{ij}(|\mathbf{x}| \mathbf{e}_\alpha) - \mathbf{H}_{ij}(0)}{|\mathbf{x}|} \mathbf{H}_{ij, \alpha}(|\mathbf{x}| \mathbf{e}_\alpha) = \lim_{|\mathbf{x}| \rightarrow 0} \left[\frac{h(|\mathbf{x}|) - h(0)}{|\mathbf{x}|} \right] h'(|\mathbf{x}|) \quad (54)$$

which implies $\mathbf{H}_{ij, \alpha}(0) \mathbf{H}_{ij, \alpha}(0) = \left[h'(0) \right]^2 = 0$ for each $\alpha = 1 \dots 3$. Hence, $\nabla \mathbf{H}_{ij}(0) = 0$. \square

Proof of (ii): The proof of (ii) can be found in [15] and [14].

Proof of (iii): We start with the relation (53) and differentiate both sides with respect to the coordinate direction \mathbf{e}_α to find

$$h'(|\mathbf{x}|) = \frac{\mathbf{H}_{ij} \mathbf{H}_{ij, \alpha} \mathbf{x}_\alpha}{h |\mathbf{x}|}. \quad (55)$$

Here $\left| \frac{\mathbf{x}_\alpha}{|\mathbf{x}|} \right| \leq 1$ and $\left| \frac{\mathbf{H}_{ij}}{h} \right| = 1$. From (55), we have that

$$\left| h'(|\mathbf{x}|) \right| \leq C_1 \|\nabla \mathbf{H}\|_{L^\infty(\mathbb{R}^3)} \quad (56)$$

where $C_1 > 0$ is independent of t . From the gradient bound derived in *Lemma A.1* in [2] and standard results in elliptic regularity (also refer to Proposition 5 (vi)), we have that $\|\nabla \mathbf{H}\|_{L^\infty(\mathbb{R}^3)}$ and all higher derivatives of the radial-hedgehog solution \mathbf{H} can be bounded independently of t in the $t \rightarrow \infty$ limit. The inequality (50) now follows. \square

Proof of (iv): We start with the relation (53) and differentiate twice to obtain

$$\mathbf{H}_{ij} \mathbf{H}_{ij,\alpha\beta} + \mathbf{H}_{ij,\alpha} \mathbf{H}_{ij,\beta} = \frac{\partial h}{\partial \mathbf{x}_\alpha} \frac{\partial h}{\partial \mathbf{x}_\beta} + h \frac{\partial^2 h}{\partial \mathbf{x}_\alpha \partial \mathbf{x}_\beta}. \quad (57)$$

A direct computation (see (46)) shows that $\mathbf{H}_{ij,\alpha} \mathbf{H}_{ij,\beta} = \frac{\partial h}{\partial \mathbf{x}_\alpha} \frac{\partial h}{\partial \mathbf{x}_\beta}$ so that

$$\frac{\partial^2 h}{\partial \mathbf{x}_\alpha \partial \mathbf{x}_\beta} = \frac{\mathbf{H}_{ij} \mathbf{H}_{ij,\alpha\beta}}{h} \quad (58)$$

and hence

$$\left| \frac{\partial^2 h}{\partial \mathbf{x}_\alpha \partial \mathbf{x}_\beta} \right| \leq C_3 \|\nabla^2 \mathbf{H}\|_{L^\infty(\mathbb{R}^3)} \leq C_4$$

for positive constants C_3 and C_4 independent of t in the $t \rightarrow \infty$ limit. Finally, it suffices to note that

$$h''(|\mathbf{x}|) = \frac{\partial^2 h}{\partial \mathbf{x}_\alpha \partial \mathbf{x}_\beta} \frac{\mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2}$$

and (51) now follows. \square

Proof of (v): We compute an explicit expression for $\frac{\partial^3 h}{\partial \mathbf{x}_\alpha \partial \mathbf{x}_\beta \partial \mathbf{x}_\gamma}$ as shown below -

$$\begin{aligned} \frac{\partial^3 h}{\partial \mathbf{x}_\alpha \partial \mathbf{x}_\beta \partial \mathbf{x}_\gamma} &= h'''(|\mathbf{x}|) \frac{\mathbf{x}_\alpha \mathbf{x}_\beta \mathbf{x}_\gamma}{|\mathbf{x}|^3} + h''(|\mathbf{x}|) \left[\frac{\mathbf{x}_\beta \delta_{\alpha\gamma} + \mathbf{x}_\alpha \delta_{\beta\gamma}}{|\mathbf{x}|^2} - \frac{2\mathbf{x}_\alpha \mathbf{x}_\beta \mathbf{x}_\gamma}{|\mathbf{x}|^4} \right] + \\ &+ h''(|\mathbf{x}|) \frac{\mathbf{x}_\gamma}{|\mathbf{x}|} \left[\frac{\delta_{\alpha\beta}}{|\mathbf{x}|} - \frac{\mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^3} \right] + h'(|\mathbf{x}|) \left[-\frac{\delta_{\alpha\beta} \mathbf{x}_\gamma + \delta_{\alpha\gamma} \mathbf{x}_\beta + \delta_{\beta\gamma} \mathbf{x}_\alpha}{|\mathbf{x}|^3} + \frac{3\mathbf{x}_\alpha \mathbf{x}_\beta \mathbf{x}_\gamma}{|\mathbf{x}|^5} \right]. \end{aligned} \quad (59)$$

We multiply both sides of (59) by $\frac{\mathbf{x}_\alpha \mathbf{x}_\beta \mathbf{x}_\gamma}{|\mathbf{x}|^3}$ to obtain

$$h'''(|\mathbf{x}|) = \frac{\partial^3 h}{\partial \mathbf{x}_\alpha \partial \mathbf{x}_\beta \partial \mathbf{x}_\gamma} \frac{\mathbf{x}_\alpha \mathbf{x}_\beta \mathbf{x}_\gamma}{|\mathbf{x}|^3}. \quad (60)$$

Straightforward computations show that

$$\frac{\partial^3 h}{\partial \mathbf{x}_\alpha \partial \mathbf{x}_\beta \partial \mathbf{x}_\gamma} = \frac{\mathbf{H}_{ij} \mathbf{H}_{ij,\alpha\beta\gamma}}{h} \quad (61)$$

and on combining (60) and (61), we obtain

$$h'''(|\mathbf{x}|) = \frac{\mathbf{H}_{ij} \mathbf{H}_{ij,\alpha\beta\gamma}}{h} \frac{\mathbf{x}_\alpha \mathbf{x}_\beta \mathbf{x}_\gamma}{|\mathbf{x}|^3}. \quad (62)$$

The inequality (52) now follows. \square

Lemma 2 Assume that \mathbf{Q}^t is a global uniaxial Landau-de Gennes minimizer in the admissible space $\mathcal{A}_{\mathbf{Q}}$ defined in (31) with $\mathbf{Q}^t(0) = 0$. Then $\nabla \mathbf{Q}^t(0) = 0$.

Proof: From the results in [11], we have that a uniaxial Landau-de Gennes minimizer can be written in the form

$$\mathbf{Q}^t = \sqrt{\frac{3}{2}} |\mathbf{Q}^t| \left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3} \mathbf{I} \right)$$

for some $\mathbf{n} \in W^{1,2}(B(0, R_t), S^2)$. Therefore, $|\mathbf{Q}^t|$ has a minimum at the origin and we have $\nabla |\mathbf{Q}^t|(0) = 0$. Using the relation $\mathbf{Q}_{ij}^t \mathbf{Q}_{ij}^t = |\mathbf{Q}^t|^2$, $\mathbf{Q}^t(0) = 0$ and $\nabla |\mathbf{Q}^t|(0) = 0$, we can repeat the same steps as in Theorem 4 (i) (see (53) and (54)) to deduce that $\nabla \mathbf{Q}^t(0) = 0$. \square

Proposition 6 Assume that \mathbf{Q}^t is a global uniaxial Landau-de Gennes minimizer in the space $\mathcal{A}_{\mathbf{Q}}$ for t sufficiently large, so that Proposition 5 holds. Define

$$\mathbf{S}_{ij}(\mathbf{x}) = \frac{\mathbf{Q}_{ij}^t(\mathbf{x})}{h(|\mathbf{x}|)} \quad i, j = 1 \dots 3. \quad (63)$$

Then

$$|\mathbf{S}(\mathbf{x})| \leq C_7 \quad \forall \mathbf{x} \in B(0, R_t) \quad (64)$$

where C_7 is a positive constant independent of t . We have the following bounds for $\nabla \mathbf{S}$:

$$|\nabla \mathbf{S}(\mathbf{x})| \leq \begin{cases} \frac{C_9}{|\mathbf{x}|} & |\mathbf{x}|^2 \leq 1 \\ C_{10} & |\mathbf{x}|^2 > 1 \end{cases} \quad (65)$$

where C_9 and C_{10} are positive constants independent of t .

Proof: From Theorem 4, we have

$$h(|\mathbf{x}|) \geq \frac{|\mathbf{x}|^2}{15} \quad |\mathbf{x}|^2 \leq 1 \quad (66)$$

so that

$$|\mathbf{S}| = \left| \frac{\mathbf{Q}^t}{h} \right| \leq \frac{15 |\mathbf{Q}^t|}{|\mathbf{x}|^2}$$

where $\|\mathbf{Q}^t\|_{L^\infty(\mathbb{R}^3)} \leq 1$, $\mathbf{Q}^t(0) = 0$ and $\nabla \mathbf{Q}^t(0) = 0$ (see Lemma 2). From standard Taylor expansion formulae [?], we have

$$\mathbf{Q}_{ij}^t(\mathbf{x}) = \int_0^1 \mathbf{Q}_{ij, \alpha\beta}^t(s\mathbf{x}) \mathbf{x}_\alpha \mathbf{x}_\beta (1-s) ds \quad |\mathbf{x}|^2 \leq 1 \quad (67)$$

so that

$$|\mathbf{Q}_{ij}^t(\mathbf{x})| \leq \|\nabla^2 \mathbf{Q}^t\|_{L^\infty(\mathbb{R}^3)} |\mathbf{x}|^2. \quad (68)$$

\mathbf{Q}^t is a classical solution of (34) and from [2] and standard results in elliptic regularity [5] (also see Proposition 5 (vi)), we have that $\|\nabla \mathbf{Q}^t\|_{L^\infty}$, $\|\nabla^2 \mathbf{Q}^t\|_{L^\infty}$ and all higher derivatives of \mathbf{Q}^t can be bounded independently of t in the $t \rightarrow \infty$ limit. Substituting the bounds (66) and (68) into the definition of \mathbf{S} , we obtain

$$|\mathbf{S}(\mathbf{x})| \leq C_{11} \quad |\mathbf{x}|^2 \leq 1 \quad (69)$$

where C_{11} is a positive constant independent of t . For $|\mathbf{x}|^2 \geq 1$, it suffices to note that $\|\mathbf{Q}^t\|_{L^\infty(\mathbb{R}^3)} \leq 1$ and $h(|\mathbf{x}|) \geq \frac{1}{15}$ for $|\mathbf{x}|^2 \geq 1$. Hence, $|\mathbf{S}(\mathbf{x})| \leq C_{12}$ for $|\mathbf{x}|^2 \geq 1$ where C_{12} is a positive constant independent of t in the $t \rightarrow \infty$ limit. The inequality (64) now follows.

Next, we compute bounds for $\nabla \mathbf{S}$. Consider the case $|\mathbf{x}|^2 \leq 1$ first. We re-write \mathbf{S} as

$$\mathbf{S} = \frac{\mathbf{Q}^t/|\mathbf{x}|^2}{h(|\mathbf{x}|)/|\mathbf{x}|^2} \quad (70)$$

so that

$$|\nabla \mathbf{S}| \leq D \left[\frac{|\nabla (\mathbf{Q}^t/|\mathbf{x}|^2)|}{h(|\mathbf{x}|)/|\mathbf{x}|^2} + \frac{|\mathbf{Q}^t/|\mathbf{x}|^2|}{(h(|\mathbf{x}|)/|\mathbf{x}|^2)^2} |\nabla (h(|\mathbf{x}|)/|\mathbf{x}|^2)| \right] \quad (71)$$

where $D > 0$ is a constant independent of t . We start with the integral formula (67) and compute $\nabla (\mathbf{Q}^t/|\mathbf{x}|^2)$. Straightforward but tedious computations show that

$$\left| \nabla \frac{\mathbf{Q}^t}{|\mathbf{x}|^2} \right| \leq \frac{D_1}{|\mathbf{x}|} \quad |\mathbf{x}|^2 \leq 1 \quad (72)$$

where $D_1 > 0$ is independent of t . Similarly, we have for $|\mathbf{x}|^2 \leq 1$ (see Theorem 4),

$$h(|\mathbf{x}|) = \int_0^{|\mathbf{x}|} h''(s) (|\mathbf{x}| - s) ds \quad (73)$$

so that

$$\left| \nabla \frac{h(|\mathbf{x}|)}{|\mathbf{x}|^2} \right| \leq \frac{D_2}{|\mathbf{x}|} \quad |\mathbf{x}|^2 \leq 1 \quad (74)$$

where $D_2 > 0$ is independent of t . From the bounds in Theorem 4 and (68), we have that $\frac{\mathbf{Q}^t}{|\mathbf{x}|^2}$ and $\frac{h(|\mathbf{x}|)}{|\mathbf{x}|^2}$ can be bounded independently of t for $|\mathbf{x}|^2 \leq 1$. Substituting (72) and (74) into (71), we have

$$|\nabla \mathbf{S}(\mathbf{x})| \leq \frac{D_3}{|\mathbf{x}|} \quad |\mathbf{x}|^2 \leq 1.$$

For $|\mathbf{x}|^2 \geq 1$, we have

$$\frac{\partial \mathbf{S}_{ij}}{\partial \mathbf{x}_\gamma} = \frac{\mathbf{Q}_{ij,\gamma}^t}{h} - \frac{\mathbf{Q}_{ij}^t}{h^2} h'(|\mathbf{x}|) \frac{\mathbf{x}_\gamma}{|\mathbf{x}|}.$$

For $|\mathbf{x}|^2 \geq 1$, h is bounded away from zero, $\|\mathbf{Q}^t\|_{L^\infty(\mathbb{R}^3)} \leq 1$, $\|\nabla \mathbf{Q}^t\|_{L^\infty(\mathbb{R}^3)}$ and $\|h'(|\mathbf{x}|)\|_{L^\infty(\mathbb{R}^3)}$ can be bounded independently of t . Hence

$$|\nabla \mathbf{S}| \leq D_4 \quad |\mathbf{x}|^2 \geq 1$$

for a constant $D_4 > 0$ independent of t . The bounds (65) now follow. \square

We need more careful estimates for \mathbf{S} and its gradient near the origin for the subsequent analysis.

Proposition 7 *Let \mathbf{Q}^t denote a global uniaxial Landau-de Gennes minimizer in the space $\mathcal{A}_{\mathbf{Q}}$ for t sufficiently large, so that Proposition 5 holds. Define \mathbf{S} as in (63). Then*

$$\begin{aligned} \mathbf{S}_{ij}(\mathbf{x}) &= \mathbf{B}_{ij\alpha\beta} \frac{\mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} + o(1) \quad |\mathbf{x}| \rightarrow 0 \\ \frac{\partial \mathbf{S}_{ij}}{\partial \mathbf{x}_\gamma} &= \frac{\partial}{\partial \mathbf{x}_\gamma} \left[\mathbf{B}_{ij\alpha\beta} \frac{\mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right] + O(1) \quad |\mathbf{x}| \rightarrow 0 \end{aligned} \quad (75)$$

where $\mathbf{B}_{ij\alpha\beta} = \frac{\mathbf{Q}_{ij,\alpha\beta}^t(0)}{h''(0)}$, $i, j, \alpha, \beta = 1 \dots 3$ is a constant matrix.

Proof: The proof of Proposition 7 follows from lengthy and technical computations. We state the main points for completeness. An explicit computation shows that

$$\frac{\partial \mathbf{S}_{ij}}{\partial \mathbf{x}_\gamma} = \frac{\partial}{\partial \mathbf{x}_\gamma} \left[\mathbf{B}_{ij\alpha\beta} \frac{\mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right] - A_{ij\gamma}^*(\mathbf{Q}^t; h) + B_{ij\gamma}^*(\mathbf{Q}^t; h) + C_{ij\gamma}^*(\mathbf{Q}^t; h) \quad (76)$$

where

$$\begin{aligned} A_{ij\gamma}^*(\mathbf{Q}^t; h) &= \frac{\mathbf{Q}_{ij}^t / |\mathbf{x}|^2}{(h(|\mathbf{x}|) / |\mathbf{x}|^2)^2} \frac{\partial}{\partial \mathbf{x}_\gamma} \left(\frac{h(|\mathbf{x}|)}{|\mathbf{x}|^2} \right) \\ B_{ij\gamma}^*(\mathbf{Q}^t; h) &= \frac{|\mathbf{x}|^2}{h(|\mathbf{x}|)} \frac{\partial}{\partial \mathbf{x}_\gamma} \left[\frac{\mathbf{Q}_{ij}^t}{|\mathbf{x}|^2} - \mathbf{Q}_{ij,\alpha\beta}^t(0) \frac{\mathbf{x}_\alpha \mathbf{x}_\beta}{2|\mathbf{x}|^2} \right] \\ C_{ij\gamma}^*(\mathbf{Q}^t; h) &= \frac{\partial}{\partial \mathbf{x}_\gamma} \left[\mathbf{Q}_{ij,\alpha\beta}^t(0) \frac{\mathbf{x}_\alpha \mathbf{x}_\beta}{2|\mathbf{x}|^2} \right] \left[\frac{|\mathbf{x}|^2}{h(|\mathbf{x}|)} - \frac{2}{h''(0)} \right]. \end{aligned} \quad (77)$$

The next step is to show that A^*, B^*, C^* are bounded as $|\mathbf{x}| \rightarrow 0$ in the $t \rightarrow \infty$ limit.

We recall that

$$\frac{h}{|\mathbf{x}|^2} \geq \frac{1}{15}$$

for $|\mathbf{x}|^2 \leq 1$ (see Theorem 4) and

$$\left| \frac{\mathbf{Q}^t(\mathbf{x})}{|\mathbf{x}|^2} \right| \leq D_5 \|\nabla^2 \mathbf{Q}^t\|_{L^\infty(\mathbb{R}^3)}$$

where D_5 is a constant independent of t (see (68)). A straightforward computation shows that

$$h(|\mathbf{x}|) = h''(0) \frac{|\mathbf{x}|^2}{2} + \int_0^{|\mathbf{x}|} h'''(\tau) \frac{(|\mathbf{x}| - \tau)^2}{2} d\tau \quad (78)$$

so that

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}_\gamma} \left(\frac{h(|\mathbf{x}|)}{|\mathbf{x}|^2} \right) &= \frac{\partial}{\partial |\mathbf{x}|} \left(\frac{h(|\mathbf{x}|)}{|\mathbf{x}|^2} \right) \frac{\mathbf{x}_\gamma}{|\mathbf{x}|} = \\ &= \frac{\mathbf{x}_\gamma}{|\mathbf{x}|} \int_0^{|\mathbf{x}|} h'''(\tau) \left(1 - \frac{\tau}{|\mathbf{x}|} \right) \frac{\tau}{|\mathbf{x}|^2} d\tau. \end{aligned} \quad (79)$$

The equality (79) implies that

$$\left| \frac{\partial}{\partial \mathbf{x}_\gamma} \left(\frac{h(|\mathbf{x}|)}{|\mathbf{x}|^2} \right) \right| \leq D_6 \|h'''\|_{L^\infty} \quad (80)$$

and from (52), we have

$$\left| \frac{\partial}{\partial \mathbf{x}_\gamma} \left(\frac{h(|\mathbf{x}|)}{|\mathbf{x}|^2} \right) \right| \leq D_7 \quad (81)$$

where D_6, D_7 are positive constants independent of t . The inequality (81) immediately implies that

$$|A_{ij\gamma}^*(\mathbf{Q}^t; h)| \leq D_8 \quad (82)$$

where $D_8 > 0$ is independent of t .

Next, we turn to $B_{ij\gamma}^*(\mathbf{Q}^t; h)$. Recalling that $\mathbf{Q}_{ij}^t(0) = \nabla \mathbf{Q}_{ij}^t(0) = 0$, we have

$$\frac{\mathbf{Q}_{ij}^t(\mathbf{x})}{|\mathbf{x}|^2} = \frac{\mathbf{Q}_{ij,\alpha\beta}^t(0)\mathbf{x}_\alpha\mathbf{x}_\beta}{2|\mathbf{x}|^2} + \int_0^1 \mathbf{Q}_{ij,\alpha\beta\sigma}^t(s\mathbf{x}) \frac{\mathbf{x}_\alpha\mathbf{x}_\beta\mathbf{x}_\sigma}{2|\mathbf{x}|^2} (1-s)^2 ds. \quad (83)$$

Therefore,

$$\frac{h(|\mathbf{x}|)}{|\mathbf{x}|^2} B_{ij\gamma}^*(\mathbf{Q}^t; h) = \int_0^1 (1-s)^2 \left[\mathbf{Q}_{ij,\alpha\beta\sigma\gamma}^t(s\mathbf{x}) s \frac{\mathbf{x}_\alpha\mathbf{x}_\beta\mathbf{x}_\sigma}{2|\mathbf{x}|^2} + \mathbf{Q}_{ij,\alpha\beta\sigma}^t(s\mathbf{x}) \frac{\partial}{\partial \mathbf{x}_\gamma} \left(\frac{\mathbf{x}_\alpha\mathbf{x}_\beta\mathbf{x}_\sigma}{2|\mathbf{x}|^2} \right) \right] ds. \quad (84)$$

Hence,

$$|B_{ij\gamma}^*(\mathbf{Q}^t; h)| \leq D_9 [\|\nabla^4 \mathbf{Q}^t\|_{L^\infty(\mathbb{R}^3)} |\mathbf{x}| + 1] \leq D_{10} \quad (85)$$

where D_9 and D_{10} are positive constants independent of t .

Finally, we turn to $C_{ij\gamma}^*(\mathbf{Q}^t; h)$. From (78), we obtain the following inequality

$$|\mathbf{x}|^2 h''(0) - 2h(|\mathbf{x}|) \leq D_{11} \|h'''\|_{L^\infty(\mathbb{R}^3)} |\mathbf{x}|^3 \quad (86)$$

which when combined with the inequalities

$$\left| \frac{\partial}{\partial \mathbf{x}_\gamma} \left(\frac{\mathbf{x}_\alpha\mathbf{x}_\beta}{|\mathbf{x}|^2} \right) \right| \leq \frac{D_{12}}{|\mathbf{x}|} \quad \text{and} \quad h(|\mathbf{x}|) \geq \frac{|\mathbf{x}|^2}{15} \quad \text{for } |\mathbf{x}|^2 \leq 1$$

yields

$$|C_{ij\gamma}^*(\mathbf{Q}^t; h)| \leq D_{13} \quad (87)$$

where D_{11}, D_{12} and D_{13} are positive constants independent of t ¹.

From (82), (85) and (87), we deduce that

$$\frac{\partial \mathbf{S}_{ij}}{\partial \mathbf{x}_\gamma} = \frac{\partial}{\partial \mathbf{x}_\gamma} \left[\mathbf{B}_{ij\alpha\beta} \frac{\mathbf{x}_\alpha\mathbf{x}_\beta}{|\mathbf{x}|^2} \right] + O(1) \quad \text{as } |\mathbf{x}| \rightarrow 0 \text{ and } t \rightarrow \infty. \quad (88)$$

Next, we consider an explicit Taylor expansion for \mathbf{S}_{ij} near the origin:

$$\mathbf{S}_{ij}(\mathbf{x}) = B_{ij\alpha\beta} \frac{\mathbf{x}_\alpha\mathbf{x}_\beta}{|\mathbf{x}|^2} + B_{ij\alpha\beta} \frac{\mathbf{x}_\alpha\mathbf{x}_\beta}{2h(|\mathbf{x}|)|\mathbf{x}|^2} \left[|\mathbf{x}|^2 h''(0) - 2h(|\mathbf{x}|) \right] + \frac{1}{2h} \int_0^1 (1-s)^2 \mathbf{Q}_{ij,\alpha\beta\gamma}^t(s\mathbf{x}) \mathbf{x}_\alpha\mathbf{x}_\beta\mathbf{x}_\gamma ds. \quad (89)$$

From (86), we obtain

$$\left| B_{ij\alpha\beta} \frac{\mathbf{x}_\alpha\mathbf{x}_\beta}{2h(|\mathbf{x}|)|\mathbf{x}|^2} \left[h''(0) - 2h(|\mathbf{x}|) \right] \right| \leq D_{14} \|\nabla^2 \mathbf{Q}^t\|_{L^\infty(\mathbb{R}^3)} |\mathbf{x}| \leq D_{15} |\mathbf{x}|$$

and

$$\left| \frac{1}{2h} \int_0^1 (1-s)^2 \mathbf{Q}_{ij,\alpha\beta\gamma}^t(s\mathbf{x}) \mathbf{x}_\alpha\mathbf{x}_\beta\mathbf{x}_\gamma ds \right| \leq D_{16} \|\nabla^3 \mathbf{Q}^t\|_{L^\infty(\mathbb{R}^3)} |\mathbf{x}| \leq D_{17} |\mathbf{x}|$$

where $D_{14} - D_{17}$ are positive constants independent of t . Hence,

$$\left| \mathbf{S}_{ij}(\mathbf{x}) - B_{ij\alpha\beta} \frac{\mathbf{x}_\alpha\mathbf{x}_\beta}{|\mathbf{x}|^2} \right| \leq D_{18} |\mathbf{x}| \quad |\mathbf{x}| \rightarrow 0 \quad (90)$$

for a positive constant D_{18} independent of t . Proposition 7 now follows. \square

¹From Theorem 4, $h''(0)$ is a constant independent of t for t sufficiently large.

Theorem 5 Assume that there exists a global uniaxial Landau-de Gennes minimizer $\mathbf{Q}^t \in \mathcal{A}_{\mathbf{Q}}$ for each $t > 0$. Let $\{t_k\}$ be a sequence such that $t_k \rightarrow \infty$ as $k \rightarrow \infty$ so that (up to a subsequence), $\mathbf{Q}^{t_k} \rightarrow \mathbf{Q}^0$ in $W^{1,2}(B(0, R_{t_k}), S_0)$ as $t_k \rightarrow \infty$, where $R_k \propto \sqrt{t_k}$ and \mathbf{Q}^0 is the unique limiting harmonic map defined in (38). Define

$$(\mathbf{S}_k)_{ij}(\mathbf{x}) = \frac{\mathbf{Q}_{ij}^{t_k}(\mathbf{x})}{h(|\mathbf{x}|)} \quad i, j = 1 \dots 3 \quad (91)$$

as in (63), where $h : [0, \infty) \rightarrow [0, 1)$ has been defined in (44) - (45). Then in the limit $k \rightarrow \infty$, $\mathbf{S}_k \in C^2(\mathbb{R}^3 \setminus \{0\}; S_0)$,

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{R_k} \int_{B(0, R_k)} \frac{1}{2} |\nabla \mathbf{S}_k|^2 + \frac{(1 - |\mathbf{S}_k|^2)^2}{4} dV &\leq 12\pi \\ \lim_{k \rightarrow \infty} \frac{1}{R_k} \int_{B(0, R_k)} \frac{|1 - |\mathbf{S}_k|^2|}{|\mathbf{x}|^2} dV &= 0. \end{aligned} \quad (92)$$

Proof: In what follows, we drop the subscript k for brevity and work with t_k large enough so that Propositions 5, 6 and 7 hold. From global energy minimality and the inequality established in Lemma 1, we have (see Proposition 5)

$$\frac{1}{R_t} \int_{B(0, R_t)} \frac{1}{2} |\nabla \mathbf{Q}^t|^2 + \frac{(1 - |\mathbf{Q}^t|^2)^2}{4} dV \leq \frac{1}{R_t} \int_{B(0, R_t)} \frac{1}{2} |\nabla \mathbf{Q}^t|^2 + f(\mathbf{Q}^t) dV \leq 12\pi \quad (93)$$

where $R_t \propto \sqrt{t}$ and $f(\mathbf{Q}^t)$ has been defined in Lemma 1. From the strong convergence in Proposition 1 and the inequality established in Lemma 1, we have

$$\lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} \frac{(1 - |\mathbf{Q}^t|^2)^2}{4} dV \leq \lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} f(\mathbf{Q}^t) dV = 0. \quad (94)$$

We note that

$$|1 - |\mathbf{S}|^2|^2 \leq 2 \left(\left| 1 - \frac{1}{h^2} \right|^2 + \frac{|1 - |\mathbf{Q}^t|^2|^2}{h^4} \right). \quad (95)$$

For $|\mathbf{x}|^2 \leq 1$, $|\mathbf{S}| \leq G_1$ for a positive constant G_1 independent of t ; see Proposition 6. For $|\mathbf{x}|^2 \geq 1$, we recall the bounds in Theorem 4 to find

$$\begin{aligned} \left| 1 - \frac{1}{h^2} \right|^2 &\leq \frac{\delta_1}{|\mathbf{x}|^4} \quad |\mathbf{x}|^2 \geq 1 \\ |1 - |\mathbf{S}|^2|^2 &\leq \delta_2 |1 - |\mathbf{Q}^t|^2|^2 \quad |\mathbf{x}|^2 \geq 1 \end{aligned} \quad (96)$$

where $\delta_1, \delta_2 > 0$ are independent of t . Combining the inequalities (95) and (96), we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} |1 - |\mathbf{S}|^2|^2 dV \leq \delta_3 \lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} (1 - |\mathbf{Q}^t|^2)^2 dV = 0 \quad (97)$$

where $\delta_3 > 0$ is independent of t .

Next, we turn to the elastic term, $|\nabla \mathbf{S}|^2$. For $|\mathbf{x}|^2 \leq r_0^2$ where $r_0 \gg 1$ is a constant independent of t , we use the estimates in Proposition 6 to obtain

$$\int_{B(0, r_0)} \frac{1}{2} |\nabla \mathbf{S}|^2 dV \leq \delta_4 r_0^3 \quad (98)$$

where $\delta_4 > 0$ is independent of t . On the region $B(0, R_t) \setminus B(0, r_0)$, an explicit computation shows that

$$|\nabla \mathbf{S}|^2 = \frac{|\nabla \mathbf{Q}^t|^2}{h^2} + |\mathbf{Q}^t|^2 \left(\frac{h'}{h^2} \right)^2 - 2\mathbf{Q}_{ij}^t \mathbf{Q}_{ij,k}^t \frac{\mathbf{x}_k h'}{|\mathbf{x}|^3} \quad i, j, k = 1 \dots 3. \quad (99)$$

For $|\mathbf{x}| \geq r_0$, we recall from [14] that

$$\left| h'(|\mathbf{x}|) \right| \leq \frac{\delta_5}{|\mathbf{x}|^3} \quad (100)$$

for a positive constant δ_5 independent of t and

$$\frac{|\nabla \mathbf{S}|^2}{h^2} = |\nabla \mathbf{Q}^t|^2 \left(1 + \frac{\delta_6}{|\mathbf{x}|^2} \right)$$

for a positive constant $\delta_6 > 0$ independent of t , in this regime. Combining (97) - (100), we deduce the following chain of inequalities

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} \frac{1}{2} |\nabla \mathbf{S}|^2 + \frac{|1 - |\mathbf{S}|^2|^2}{4} dV &= \lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} \frac{1}{2} |\nabla \mathbf{S}|^2 dV \leq \\ &\leq \lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} \frac{1}{2} |\nabla \mathbf{Q}^t|^2 dV \leq 12\pi \end{aligned} \quad (101)$$

where the last inequality follows from Proposition 5.

Finally, we turn to the integral $\lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} \frac{|1 - |\mathbf{S}|^2|}{|\mathbf{x}|^2} dV$. Recall that $|\mathbf{S}|$ satisfies the global upper bound $|\mathbf{S}| \leq \delta_7$ for a positive constant δ_7 independent of t (see Proposition 6). Then a direct computation shows that

$$\int_{B(0, 1)} \frac{|1 - |\mathbf{S}|^2|}{|\mathbf{x}|^2} dV \leq \delta_8 \quad (102)$$

for a positive constant δ_8 independent of t . On the region $B(0, R_t) \setminus B(0, 1)$, we use Young's inequality to deduce

$$\frac{|1 - |\mathbf{S}|^2|}{|\mathbf{x}|^2} \leq \frac{1}{2} \left[\frac{(1 - |\mathbf{S}|^2)^2}{4} + \frac{4}{|\mathbf{x}|^4} \right]. \quad (103)$$

Combining (102)-(103) and recalling (97), we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} \frac{|1 - |\mathbf{S}|^2|}{|\mathbf{x}|^2} dV &\leq \\ &\leq \lim_{t \rightarrow \infty} \left[\frac{\delta_8}{R_t} + \frac{1}{2R_t} \int_{B(0, R_t) \setminus B(0, 1)} \frac{(1 - |\mathbf{S}|^2)^2}{4} dV + \frac{1}{2R_t} \int_{B(0, R_t) \setminus B(0, 1)} \frac{4}{|\mathbf{x}|^4} dV \right] = 0 \end{aligned} \quad (104)$$

as required. The proof of Theorem 5 is now complete. \square

3.1 PDE-methods for \mathbf{S}

Let $\mathbf{Q}^t \in \mathcal{A}_{\mathbf{Q}}$ denote a uniaxial global Landau-de Gennes minimizer for t sufficiently large so that Propositions 5, 6, 7 and Theorem 5 hold. Define \mathbf{S} as in (63). We recall that \mathbf{Q}^t is a classical solution of

$$\Delta \mathbf{Q}_{ij}^t = \left[(|\mathbf{Q}^t|^2 - 1) + \frac{3h_+}{t} (|\mathbf{Q}^t|^2 - |\mathbf{Q}^t|) \right] \mathbf{Q}_{ij}^t \quad (105)$$

and h is a solution of the ordinary differential equation

$$h'' + \frac{2}{|\mathbf{x}|}h' - \frac{6h}{|\mathbf{x}|^2} = h^3 - h + \frac{3h_+}{t}(h^3 - h^2) \quad (106)$$

where h' denotes $\frac{dh}{d|\mathbf{x}|}$ etc. Straightforward but lengthy manipulations show that \mathbf{S} satisfies the following system of partial differential equations -

$$\Delta \mathbf{S}_{ij} + \left(1 + \frac{3h_+}{t}\right) h^2 (1 - |\mathbf{S}|^2) \mathbf{S}_{ij} = -2 \frac{h'}{h} \mathbf{S}_{ij,k} \frac{\mathbf{x}_k}{|\mathbf{x}|} - \frac{6\mathbf{S}_{ij}}{|\mathbf{x}|^2} + \frac{3h_+}{t} h \mathbf{S}_{ij} (1 - |\mathbf{S}|) \quad i, j = 1 \dots 3. \quad (107)$$

Following the methods in [18], we multiply both sides of the partial differential equation (107) with $\mathbf{S}_{ij,k} \frac{\mathbf{x}_k}{|\mathbf{x}|}$. One can check that

$$\mathbf{S}_{ij,k} \frac{\mathbf{x}_k}{|\mathbf{x}|} \Delta \mathbf{S}_{ij} = \frac{1}{|\mathbf{x}|} \left(\frac{\partial \mathbf{S}_{ij}}{\partial |\mathbf{x}|} \right)^2 + \frac{\partial}{\partial \mathbf{x}_p} \left[-\frac{1}{2} |\nabla \mathbf{S}|^2 \frac{\mathbf{x}_p}{|\mathbf{x}|} + \mathbf{S}_{ij,k} \frac{\mathbf{x}_k}{|\mathbf{x}|} \mathbf{S}_{ij,p} \right] \quad (108)$$

$$\begin{aligned} & \left(1 + \frac{3h_+}{t}\right) h^2 (|\mathbf{x}|) (1 - |\mathbf{S}|^2) \mathbf{S}_{ij} \mathbf{S}_{ij,k} \frac{\mathbf{x}_k}{|\mathbf{x}|} = \\ & = \left(1 + \frac{3h_+}{t}\right) \left[\frac{(1 - |\mathbf{S}|^2)^2}{4} \left[2hh' + \frac{2h^2}{r} \right] - \frac{\partial}{\partial \mathbf{x}_p} \left(\frac{\mathbf{x}_p}{|\mathbf{x}|} \frac{h^2 (1 - |\mathbf{S}|^2)^2}{4} \right) \right] \end{aligned} \quad (109)$$

$$-2 \frac{h'}{h} \mathbf{S}_{ij,k} \frac{\mathbf{x}_k}{|\mathbf{x}|} \mathbf{S}_{ij,p} \frac{\mathbf{x}_p}{|\mathbf{x}|} = -2 \frac{h'}{h} \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 \text{ and} \quad (110)$$

$$-6 \frac{\mathbf{S}_{ij}}{|\mathbf{x}|^2} \mathbf{S}_{ij,p} \frac{\mathbf{x}_p}{|\mathbf{x}|} = \frac{\partial}{\partial \mathbf{x}_p} \left[\frac{3\mathbf{x}_p}{|\mathbf{x}|^3} (1 - |\mathbf{S}|^2) \right]. \quad (111)$$

Using (108) - (111), we obtain

$$\begin{aligned} & \frac{\partial}{\partial \mathbf{x}_p} \left[\frac{1}{2} |\nabla \mathbf{S}|^2 \frac{\mathbf{x}_p}{|\mathbf{x}|} - \frac{\mathbf{S}_{ij,k} \mathbf{x}_k}{|\mathbf{x}|} \mathbf{S}_{ij,p} + \left(1 + \frac{3h_+}{t}\right) \frac{\mathbf{x}_p}{|\mathbf{x}|} \frac{h^2 (1 - |\mathbf{S}|^2)^2}{4} + \frac{3\mathbf{x}_p (1 - |\mathbf{S}|^2)}{|\mathbf{x}|^3} \right] = \\ & = \frac{1}{|\mathbf{x}|} \left(\frac{\partial \mathbf{S}_{ij}}{\partial |\mathbf{x}|} \right)^2 + \left(1 + \frac{3h_+}{t}\right) \frac{(1 - |\mathbf{S}|^2)^2}{4} \left[2hh' + \frac{2h^2}{r} \right] + 2 \frac{h'}{h} \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 - \frac{3h_+}{t} h (1 - |\mathbf{S}|) \mathbf{S}_{ij} \mathbf{S}_{ij,k} \frac{\mathbf{x}_k}{|\mathbf{x}|} \end{aligned}$$

We recall from Theorem 4 and the results in [14] that $h' > 0$ for $|\mathbf{x}| > 0$, for t sufficiently large.

Define

$$\Phi_p = \frac{1}{2} |\nabla \mathbf{S}|^2 \frac{\mathbf{x}_p}{|\mathbf{x}|} - \frac{\mathbf{S}_{ij,k} \mathbf{x}_k}{|\mathbf{x}|} \mathbf{S}_{ij,p} + \left(1 + \frac{3h_+}{t}\right) \frac{\mathbf{x}_p}{|\mathbf{x}|} \frac{h^2 (1 - |\mathbf{S}|^2)^2}{4} + \frac{3\mathbf{x}_p (1 - |\mathbf{S}|^2)}{|\mathbf{x}|^3} \quad p = 1 \dots 3. \quad (113)$$

Lemma 3 *We have*

$$\int_{|\mathbf{x}|=\delta} \Phi_p \frac{\mathbf{x}_p}{|\mathbf{x}|} dA \rightarrow 12\pi \quad \text{as } \delta \rightarrow 0 \quad (114)$$

where dA is the surface area element on the sphere of radius δ centered at the origin.

Proof: By the definition of Φ_p in (113), we have

$$\begin{aligned} & \int_{|\mathbf{x}|=\delta} \Phi_p \frac{\mathbf{x}_p}{|\mathbf{x}|} dA = \\ & = \int_{|\mathbf{x}|=\delta} \frac{1}{2} |\nabla \mathbf{S}|^2 - \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 + \left(1 + \frac{3h_+}{t}\right) \frac{h^2 (1 - |\mathbf{S}|^2)^2}{4} + \frac{3(1 - |\mathbf{S}|^2)}{|\mathbf{x}|^2} dA. \end{aligned} \quad (115)$$

From the estimates in Proposition 7, we have the following as $\delta \rightarrow 0$

$$\begin{aligned} |\nabla \mathbf{S}|^2 &= \left| \nabla \left(\frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right) \right|^2 + o(|\mathbf{x}|^{-2}) \\ \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 &= o(|\mathbf{x}|^{-2}) \\ 1 - |\mathbf{S}|^2 &= \frac{|\mathbf{x}|^4 - |\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta|^2}{|\mathbf{x}|^4} + o(1) \end{aligned} \quad (116)$$

where $\mathbf{B}_{ij\alpha\beta} = \frac{\mathbf{Q}_{ij,\alpha\beta}^t(0)}{h''(0)}$ is a constant matrix and $i, j, \alpha, \beta = 1 \dots 3$.

Substituting (116) into (115), we get

$$\begin{aligned} \int_{|\mathbf{x}|=\delta} \Phi_p \frac{\mathbf{x}_p}{|\mathbf{x}|} dA &= \\ &= \int_{|\mathbf{x}|=\delta} \frac{1}{2} \left| \nabla \left(\frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right) \right|^2 + 3 \left(\frac{|\mathbf{x}|^4 - |\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta|^2}{|\mathbf{x}|^6} \right) + o(|\mathbf{x}|^{-2}) dA = \\ &= 12\pi + o(1) + \int_{|\mathbf{x}|=\delta} \frac{1}{2} \left| \nabla \left(\frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right) \right|^2 - \frac{3}{|\mathbf{x}|^2} \left| \frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right|^2 dA. \end{aligned} \quad (117)$$

A direct computation shows that

$$\int_{|\mathbf{x}|=\delta} \frac{1}{2} \left| \nabla \left(\frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right) \right|^2 - \frac{3}{|\mathbf{x}|^2} \left| \frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right|^2 dA = 0$$

for the constant matrix $\mathbf{B}_{ij\alpha\beta} = \frac{\mathbf{Q}_{ij,\alpha\beta}^t(0)}{h''(0)}$ and hence, the conclusion of Lemma 3 follows. \square

Lemma 4 *The integral*

$$\int_{|\mathbf{x}|=1} \frac{1}{2} \left| \nabla \left(\frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right) \right|^2 - \frac{3}{|\mathbf{x}|^2} \left| \frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right|^2 dA = 0 \quad (118)$$

for any constant $\mathbf{B}_{ij\alpha\beta}$ such that $\mathbf{B}_{ij\alpha\beta} = \mathbf{B}_{ji\alpha\beta}$, $\mathbf{B}_{ij\alpha\beta} = \mathbf{B}_{ij\beta\alpha}$ and $\mathbf{B}_{ij\alpha\alpha} = \mathbf{B}_{ii\alpha\beta} = 0$.

Proof: Consider the matrix

$$\mathbf{B}_{ij\alpha\beta} = \frac{\mathbf{Q}_{ij,\alpha\beta}^t(0)}{h''(0)} \quad i, j, \alpha, \beta = 1 \dots 3. \quad (119)$$

Recalling that \mathbf{Q}^t is a symmetric, traceless 3×3 matrix which is a classical solution the PDE-system (105) (so that we have $\mathbf{Q}_{ij,\alpha\beta}^t = \mathbf{Q}_{ij,\beta\alpha}^t$ from equality of mixed partial derivatives), we obtain

$$\mathbf{B}_{ij\alpha\beta} = \mathbf{B}_{ji\alpha\beta}; \quad \mathbf{B}_{ij\alpha\beta} = \mathbf{B}_{ij\beta\alpha}; \quad \mathbf{B}_{ij\alpha\alpha} = 0, \quad \mathbf{B}_{ii\alpha\beta} = 0 \quad (120)$$

for all $i, j, \alpha, \beta = 1 \dots 3$.

A direct computation shows that

$$\frac{1}{2} \left| \nabla \left(\frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right) \right|^2 = \frac{2}{|\mathbf{x}|^4} \mathbf{B}_{ijpq} \mathbf{B}_{ijrs} \mathbf{x}_q \mathbf{x}_s \left(\delta_{rp} - \frac{\mathbf{x}_r \mathbf{x}_p}{|\mathbf{x}|^2} \right) \quad (121)$$

and

$$\frac{3}{|\mathbf{x}|^2} \left| \frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right|^2 = \frac{3}{|\mathbf{x}|^6} \mathbf{B}_{ijpq} \mathbf{B}_{ijrs} \mathbf{x}_p \mathbf{x}_q \mathbf{x}_r \mathbf{x}_s \quad (122)$$

so that

$$\begin{aligned} & \int_{|\mathbf{x}|=1} \frac{1}{2} \left| \nabla \left(\frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right) \right|^2 - \frac{3}{|\mathbf{x}|^2} \left| \frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right|^2 dA = \\ & = \mathbf{B}_{ijpq} \mathbf{B}_{ijrs} \left[2\delta_{rp} \int_{|\mathbf{x}|=1} \mathbf{x}_q \mathbf{x}_s dA - 5 \int_{|\mathbf{x}|=1} \mathbf{x}_p \mathbf{x}_q \mathbf{x}_r \mathbf{x}_s dA \right] \end{aligned} \quad (123)$$

for $i, j, p, q, r, s = 1 \dots 3$.

Using spherical coordinate representation, we can check that

$$\int_{|\mathbf{x}|=1} \mathbf{x}_q \mathbf{x}_s dA = \frac{4\pi}{3} \delta_{qs} \quad (124)$$

and

$$\int_{|\mathbf{x}|=1} \mathbf{x}_p \mathbf{x}_q \mathbf{x}_r \mathbf{x}_s dA = \frac{4\pi}{15} [\delta_{pq} \delta_{rs} + \delta_{pr} \delta_{qs} + \delta_{ps} \delta_{qr}]. \quad (125)$$

Substituting (124) and (125) into (123), we obtain

$$\begin{aligned} & \int_{|\mathbf{x}|=1} \frac{1}{2} \left| \nabla \left(\frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right) \right|^2 - \frac{3}{|\mathbf{x}|^2} \left| \frac{\mathbf{B}_{ij\alpha\beta} \mathbf{x}_\alpha \mathbf{x}_\beta}{|\mathbf{x}|^2} \right|^2 dA = \\ & = \frac{4\pi}{3} [2\mathbf{B}_{ijrs} \mathbf{B}_{ijrs} - \mathbf{B}_{ijpp} \mathbf{B}_{ijss} - \mathbf{B}_{ijqr} \mathbf{B}_{ijrq} - \mathbf{B}_{ijsr} \mathbf{B}_{ijrs}] \end{aligned} \quad (126)$$

and the right-hand side vanishes by virtue of the properties established in (120). The integral equality (118) now follows. \square

Proposition 8 *Let $\mathbf{Q}^t \in \mathcal{A}_{\mathbf{Q}}$ denote a uniaxial global Landau-de Gennes minimizer for t sufficiently large so that Propositions 5, 6, 7 and Theorem 5 hold. Define \mathbf{S} as in (63). Then*

$$\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} = 0 \quad \mathbf{x} \in B(0, R_t) \quad (127)$$

$$|\mathbf{S}(\mathbf{x})| = 1 \quad \mathbf{x} \in B(0, R_t) \quad (128)$$

where $R_t = \mu\sqrt{t}$ and μ is a constant independent of t .

Proof: We integrate both sides of (112) from $|\mathbf{x}| = 0$ to $|\mathbf{x}| = R_t$, divide by R_t , use Lemma 3 and take limit $t \rightarrow \infty$ to obtain

$$\begin{aligned} & 12\pi + \lim_{t \rightarrow \infty} \frac{1}{R_t} \int_0^{R_t} \int_{B(0,R)} \frac{1}{|\mathbf{x}|} \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 + \left(1 + \frac{3h_+}{t} \right) \frac{(1 - |\mathbf{S}|^2)^2}{4} \left[2h' h + \frac{2h^2}{|\mathbf{x}|} \right] + \frac{2h'}{h} \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 dV dR = \\ & - \lim_{t \rightarrow \infty} \frac{3h_+}{t} \frac{1}{R_t} \int_0^{R_t} \int_{B(0,R)} h(1 - |\mathbf{S}|) \mathbf{S}_{ij} \mathbf{S}_{ij,k} \frac{\mathbf{x}_k}{|\mathbf{x}|} dV dR = \\ & = \lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0,R_t)} \frac{1}{2} |\nabla \mathbf{S}|^2 - \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 + \left(1 + \frac{3h_+}{t} \right) \frac{h^2 (1 - |\mathbf{S}|^2)^2}{4} + \frac{3(1 - |\mathbf{S}|^2)}{|\mathbf{x}|^2} dV. \end{aligned} \quad (129)$$

From (92), we have that

$$\lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} \frac{3(1 - |\mathbf{S}|^2)}{|\mathbf{x}|^2} dV = 0$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{R_t} \int_{B(0, R_t)} \frac{1}{2} |\nabla \mathbf{S}|^2 - \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 + \left(1 + \frac{3h_+}{t} \right) \frac{h^2 (1 - |\mathbf{S}|^2)^2}{4} \leq 12\pi$$

since $h^2(|\mathbf{x}|) \leq 1$ on $B(0, R_t)$ and $\lim_{t \rightarrow \infty} \frac{h_+}{t} = 0$.

From Theorem 4, we recall that h is monotonically increasing in the limit $t \rightarrow \infty$ so that every term in the integrand of

$$\frac{1}{R_t} \int_0^{R_t} \int_{B(0, R)} \frac{1}{|\mathbf{x}|} \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 + \left(1 + \frac{3h_+}{t} \right) \frac{(1 - |\mathbf{S}|^2)^2}{4} \left[2h' h + \frac{2h^2}{|\mathbf{x}|} \right] + \frac{2h'}{h} \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 dV dR$$

is non-negative.

Finally, we estimate the integral $\frac{1}{R_t} \int_0^{R_t} \int_{B(0, R)} h(1 - |\mathbf{S}|) \mathbf{S}_{ij} \mathbf{S}_{ij, k} \frac{\mathbf{x}_k}{|\mathbf{x}|} dV dR$ as follows:

$$\begin{aligned} & \left| \int_{B(0, R)} h(1 - |\mathbf{S}|) \mathbf{S}_{ij} \mathbf{S}_{ij, k} \frac{\mathbf{x}_k}{|\mathbf{x}|} dV \right| \leq \\ & \leq C \left[\int_{B(0, R)} (1 - |\mathbf{S}|^2)^2 dV \right]^{1/2} \left[\int_{B(0, R)} |\nabla \mathbf{S}|^2 dV \right]^{1/2} \leq C^* \sqrt{f[R]} \sqrt{R} \end{aligned} \quad (130)$$

where $\lim_{R \rightarrow \infty} \frac{f[R]}{R} = 0$ for all $R > 0$ where C and C^* are positive constants independent of R . The first inequality follows from Cauchy-Schwarz and the fact that $|\mathbf{S}| \leq D$ for a positive constant D independent of R (see Proposition 6). From Theorem 5, we have

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_{B(0, R)} (1 - |\mathbf{S}|^2)^2 dV = 0$$

so that if we set

$$\int_{B(0, R)} (1 - |\mathbf{S}|^2)^2 dV = f[R],$$

then $f[R] = o(R)$ as $R \rightarrow \infty$. The second inequality also uses the upper bound $\int_{B(0, R)} |\nabla \mathbf{S}|^2 dV \leq 12\pi R$ from (92), yielding the inequality (130).

Substituting the upper bound (130) into (129), we get

$$\lim_{t \rightarrow \infty} \frac{3h_+}{t} \left| \frac{1}{R_t} \int_0^{R_t} \int_{B(0, R)} h(1 - |\mathbf{S}|) \mathbf{S}_{ij} \mathbf{S}_{ij, k} \frac{\mathbf{x}_k}{|\mathbf{x}|} dV dR \right| = 0 \quad (131)$$

since $\int_0^R \sqrt{f[R]} dR = o(R^2)$ as $R \rightarrow \infty$ and $\frac{3h_+}{t} \sim \frac{1}{R_t}$ as $t \rightarrow \infty$.

Combining the above, we deduce that

$$\int_{B(0, R)} \frac{1}{|\mathbf{x}|} \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 + \left(1 + \frac{3h_+}{t} \right) \frac{(1 - |\mathbf{S}|^2)^2}{4} \left[2h' h + \frac{2h^2}{|\mathbf{x}|} \right] + \frac{2h'}{h} \left(\frac{\partial \mathbf{S}}{\partial |\mathbf{x}|} \right)^2 dV = 0 \quad (132)$$

for all $R > 0$ and the conclusion of Proposition 8 now follows. \square

Proposition 9 *Let $\mathbf{Q}^t \in \mathcal{A}_{\mathbf{Q}}$ denote a uniaxial global Landau-de Gennes minimizer for t sufficiently large so that Propositions 5, 6, 7 and Theorem 5 hold. Then in the limit $t \rightarrow \infty$, we have*

$$\mathbf{Q}^t = h(|\mathbf{x}|) \left(\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|^2} - \frac{\mathbf{I}}{3} \right) \quad \mathbf{x} \in B(0, R_t) \quad (133)$$

where h is the unique solution of (44) subject to (45) and $R_t = \mu\sqrt{t}$, with μ being a constant independent of t .

Proof: From Proposition 8, we have that in the limit $t \rightarrow \infty$

$$\mathbf{Q}^t = h(|\mathbf{x}|) \mathbf{M}_{ij} \left(\frac{\mathbf{x}}{|\mathbf{x}|} \right) \quad (134)$$

where $\mathbf{M}_{ij} = \sqrt{\frac{3}{2}} (\mathbf{m} \otimes \mathbf{m} - \frac{\mathbf{I}}{3})$ for some $\mathbf{m} \in S^2$ (from the uniaxial character of \mathbf{Q}^t),

$$|\mathbf{M}(\mathbf{x})|^2 = 1 \quad \mathbf{x} \in B(0, R_t) \quad (135)$$

and $\mathbf{M}_{ij} \rightarrow \sqrt{\frac{3}{2}} \left(\frac{\mathbf{x} \otimes \mathbf{x}}{|\mathbf{x}|} - \frac{\mathbf{I}}{3} \right)$ as $|\mathbf{x}| \rightarrow \infty$ from the imposed Dirichlet boundary condition.

We substitute (134) into the governing system of PDEs for \mathbf{Q}^t as shown below -

$$\Delta \mathbf{Q}_{ij}^t = \left[(|\mathbf{Q}^t|^2 - 1) + \frac{3h_+}{t} (|\mathbf{Q}^t|^2 - |\mathbf{Q}^t|) \right] \mathbf{Q}_{ij}^t, \quad (136)$$

multiply both sides of (136) with \mathbf{M}_{ij} (noting that $\mathbf{M}_{ij} \mathbf{M}_{ij,k} = 0$) to find

$$|\nabla \mathbf{M}|^2 = 3|\nabla \mathbf{m}|^2 = \frac{6}{r^2}. \quad (137)$$

Consider the minimization problem

$$\min_{\mathbf{n} \in N} \int_{B(0,R)} |\nabla \mathbf{n}|^2 dV \quad (138)$$

where $B(0, R)$ is a three-dimensional droplet of arbitrary radius $R > 0$ and

$$N = \left\{ \mathbf{n} \in W^{1,2}(B(0, R); S^2); \mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|} \text{ on } \partial B(0, R) \right\}. \quad (139)$$

It is known that the minimization problem (138)-(139) has a *unique* minimizer [10]

$$\mathbf{n}_{\min} = \frac{\mathbf{x}}{|\mathbf{x}|} \quad (140)$$

and

$$\int_{B(0,R)} |\nabla \mathbf{n}_{\min}|^2 dV = 8\pi R. \quad (141)$$

The unit-vector field $\mathbf{m} \in W^{1,2}(B(0, R_t); S^2)$, $\mathbf{m} = \frac{\mathbf{x}}{|\mathbf{x}|}$ on $\partial B(0, R_t)$ and from (137), we have

$$\int_{B(0,R)} |\nabla \mathbf{m}|^2 dV = 8\pi R_t. \quad (142)$$

Comparing (141) and (142), we deduce that \mathbf{m} is a minimizer of the problem (138)-(139) on $B(0, R_t)$ and from the uniqueness of the minimizer, we deduce that

$$\mathbf{m}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} \quad \mathbf{x} \in B(0, R_t).$$

The conclusion of Proposition 9 now follows. \square

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